

Exponential Stability of Linear Delay Impulsive Differential Equations

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Abstract

For an impulsive delay differential equation with bounded delay and bounded coefficients the following result is established. If each solution is bounded on $[0, \infty)$ together with its derivative for each bounded right-hand side then the equation is exponentially stable. A coefficient stability theorem is presented.

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1 Introduction

The work of Millman and Myshkis [1] was the first one dealing with impulsive differential equations. Recently this field has been intensively investigated (see the monographs [2,3] and their bibliography). Not so much has been developed in the direction of impulsive functional differential equations [4,5,6].

The paper deals with the exponential stability of a linear impulsive delay differential equation. A new method of research in stability is proposed. The stability theorem is based on two results.

The first result is the representation of solutions. The representation formulas were presented in our works [7,8]. However now we obtain the result that is more convenient for an application. The idea of its proof is also different. It is to be emphasized that a representation of solutions is exploited for the stability investigation not for a long time. The idea was developed in the works of Corduneanu [9-11].

The second result is the Bohl-Perron theorem for an impulsive equation. The Bohl-Perron theorem is the result of the following type. Suppose that any solution of the linear differential equation is bounded on $[0, \infty)$ for each bounded on $[0, \infty)$ right-hand side. Then the solution X of the corresponding homogeneous equation has an exponential estimate

$$\|X(t)\| \leq N \exp(-\nu t),$$

with positive constants N, ν .

For an ordinary differential equations without impulses Bohl-Perron theorem is discussed in detail in [12]. For a delay differential equation results of this type were obtained by Halanay [13] and Tyshkevich [14]. These results were generalized and completed in [15] and [16]. The exponential behavior of impulsive differential equations was investigated in [17,18].

We prove the Bohl-Perron theorem for impulsive delay differential equations on the base of the scheme proposed in [19]. The scheme was intensively used in the stability theory of functional differential equations [20,21].

It is to be emphasized that in the impulsive conditions

$$x(\tau_i + 0) = B_i x(\tau_i - 0)$$

we do not assume that B_i are invertible. From this point of view the result obtained is new for impulsive equations without delay as well. The point is

that the equality for the Cauchy matrix

$$C(t, s) = X(t)X^{-1}(s)$$

holds for impulsive differential equations only if B_i are invertible, and the proof of the Bohl-Perron theorem [17] is based on this equality.

In conclusion we present the exponential stability result. On the base of this theorem sufficient stability conditions for parameters of the equation are obtained.

2 Preliminaries

Let $0 = \tau_0 < \tau_1 < \dots$ be the fixed points, $\lim_{i \rightarrow \infty} \tau_i = \infty$, \mathbf{R}^n be the space of n -dimensional column vectors $x = \text{col}(x_1, \dots, x_n)$ with the norm $\|x\| = \max_{1 \leq i \leq n} |x_i|$, by the same symbol $\|\cdot\|$ we shall denote the corresponding matrix norm,

E_n is an $n \times n$ unit matrix,

$\chi_e : [0, \infty) \rightarrow \mathbf{R}$ is the characteristic function of the set $e : \chi_e(t) = 1$, if $t \in e$, and $\chi_e(t) = 0$, otherwise.

\mathbf{L}_∞ is a Banach space of essentially bounded Lebesgue measurable functions $x : [0, \infty) \rightarrow \mathbf{R}^n$, $\|x\|_{\mathbf{L}_\infty} = \text{vraisup}_{t \geq 0} \|x(t)\|$,

\mathbf{AC} is a linear space of functions $x : [0, \infty) \rightarrow \mathbf{R}^n$ absolutely continuous on any segment $[t, t+1]$, $t \geq 0$,

$\mathbf{PAC}(\tau_1, \dots, \tau_k, \dots)$ is the space of piecewise absolutely continuous functions $x : [0, \infty) \rightarrow \mathbf{R}^n$, with jumps only at the points $\tau_1, \tau_2, \dots, \tau_k, \dots$, i.e.

$$\mathbf{PAC}(\tau_1, \dots, \tau_k, \dots) = \left\{ x : [0, \infty) \rightarrow \mathbf{R}^n \mid x(t) = y(t) + \sum_{i=1}^{\infty} \alpha_i \chi_{[\tau_i, \infty)}(t), \right. \\ \left. t \geq 0, \alpha_i \in \mathbf{R}^n, y \in \mathbf{AC} \right\}.$$

It is to be noted that a function $x \in \mathbf{PAC}(\tau_1, \dots, \tau_k, \dots)$ is right continuous.

We consider the linear delay differential equation

$$\dot{x}(t) + \sum_{i=1}^m A_i(t)x[h_i(t)] = r(t), \quad t \geq 0, \quad (1)$$

$$x(\tau_j) = B_j x(\tau_j - 0), j = 1, 2, \dots, \quad (2)$$

$$x(\xi) = \varphi(\xi), \xi < 0. \quad (3)$$

Here B_j are constant $n \times n$ -matrices, $h_i(t) \leq t, i = 1, \dots, m, t \geq 0$.

We consider the equation (1),(2),(3) under the following assumptions

(a1) $0 = \tau_0 < \tau_1 < \tau_2 < \dots$ are fixed points, $\lim_{i \rightarrow \infty} \tau_i = \infty$;

(a2) $r \in \mathbf{L}_\infty$, columns of $A_i, i = 1, \dots, m$, are in \mathbf{L}_∞ ;

(a3) $h_i : [0, \infty) \rightarrow \mathbf{R}$ are Lebesgue measurable functions,

$$h_i(t) \leq t, t \geq 0, i = 1, \dots, m;$$

(a4) $\varphi : (-\infty, 0) \rightarrow \mathbf{R}^n$ is a Borel measurable bounded function;

(a5) $M = \sup_i \| B_i \| < \infty$;

(a6) $I = \lim_{b \rightarrow \infty} \sup_{t,s > b} \frac{i(t,s)}{t-s} < \infty$.

Here $i(t, s)$ is a number of points τ_j belonging to the segment $[s, t]$.

Definition 1. A function $x \in \mathbf{PAC}$ is said to be a **solution of the impulsive equation (1),(2)** with the initial function $\varphi(t)$ if (1) is satisfied for almost all $t \in [0, \infty)$ and the equalities (2) hold.

Our objective is to study the exponential stability of the impulsive equation (1),(2),(3), if any solution is bounded for a bounded on $[0, \infty)$ right-hand side r .

Definition 2. The equation (1),(2),(3) is said to be **exponentially stable** if there exist positive constants N and ν such that for any solution x of the corresponding homogeneous equation

$$\dot{x}(t) + \sum_{i=1}^m A_i(t)x[h_i(t)] = 0, t \geq 0, \quad (4)$$

(2),(3) the inequality

$$\| x(t) \| \leq N \exp(-\nu t) [\sup_{s < 0} \| \varphi(s) \| + \| x(0) \|] \quad (5)$$

holds.

If the inequality (5) is valid for $\varphi \equiv 0$, then the equation is said to be **exponentially stable with respect to the initial value**, if (5) holds for $x(0) = 0$, then the equation is said to be **exponentially stable with respect to the initial function**.

It is to be emphasized that exponential properties of delay impulsive differential equations differ greatly from the properties of the corresponding equations without delay (or without impulses). For instance in [22] the following theorem was proven in the scalar case.

Suppose that the Cauchy problem $\dot{x} + Ax = f, x(0) = 0$ has a bounded on $[0, \infty)$ solution for any $f \in \mathbf{L}_\infty$ and there exists a positive constant M_1 such that

$$|B_p B_{p+1} \dots B_j| \leq M_1$$

for any positive integers $p, j, p < j, A \in \mathbf{L}_\infty$. Then there exist positive constants N and ν such that the solution of the corresponding homogeneous equation satisfying (4),(2), $x(0) = 1$, has the exponential estimate

$$|x(t)| \leq N \exp(-\nu t).$$

The following example shows that for the delay impulsive differential equation this theorem is not valid.

Example. The equation

$$\dot{x}(t) + ax(t-1) = f, \quad 0 < a < \pi/2,$$

is exponentially stable [23]. Consider this equation with impulsive conditions

$$x(j) = -x(j-1), \quad j = 1, 2, \dots$$

Here $|B_j| = 1$. The absolute value of the solution of the corresponding homogeneous equation with $x(0) = 1$ is steadily growing, $t > 1$

$$|x(t)| = 1, \quad t \in [0, 1),$$

$$|x(t)| \geq 1 + a(t-1), \quad t \in [1, 2),$$

$$|x(t)| \geq 1 + a + a(1+a)(t-2), \quad t \in [2, 3),$$

$$|x(t)| \geq (1+a)^2 + a(1+a)^2(t-3), \quad t \in [3, 4),$$

....

3 Representation of solutions

The main result of this section deals with the representation of solutions.

In the stability theory of ordinary differential equations the representation of the Cauchy matrix

$$C(t, s) = X(t)X^{-1}(s)$$

is intensively used. Here C is the Cauchy matrix, i.e. the kernel of the integral representation of solutions, $X(t)$ is the solution of the corresponding homogeneous equation satisfying $X(0) = E_n$.

For delay differential equations this equality generally speaking is not true.

Definition 3. An impulsive delay differential equation

$$\dot{x}(t) + \sum_{i=1}^m A_i(t)x[h_i(t)] = 0, \quad t \geq s, \quad (6)$$

$$x(\xi) = 0, \quad \xi < s, \quad (7)$$

$$x(\tau_j) = B_j x(\tau_j - 0), \quad j = 1, 2, \dots, \tau_i > s,$$

is said to be a **homogeneous "s - curtailed" equation** (1),(2). The solution $X(\cdot, s)$ of this equation satisfying $X(s, s) = E_n$ is said to be **the fundamental matrix of the "s - curtailed" equation**.

The main result of this section is the following.

Theorem 3.1 *Suppose (a1)-(a4) are satisfied. Then the solution of the problem (1),(3), with the initial value*

$$x(0) = \alpha_0 \quad (8)$$

and impulsive conditions

$$x(\tau_i) = B_i x(\tau_i - 0) + \alpha_i, \quad i = 1, 2, \dots, \quad (9)$$

can be presented as

$$x(t) = \int_0^t X(t, s)r(s)ds - \sum_{i=1}^m \int_0^t X(t, s)A_i(s)\varphi(h_i(s))ds + \sum_{j=0}^{\infty} X(t, \tau_j)\alpha_j. \quad (10)$$

Here $\varphi(\zeta) = 0$, if $\zeta \geq 0$.

First we prove that there exists one and only one solution of the initial problem for the equation (1),(9),(3) (Lemma 3.1). Then we establish the coincidence of the Cauchy matrix and the fundamental matrix of "s-curtailed" equation (Lemma 3.3). To this end we need the estimate of the latter matrix (Lemma 3.2). Afterwards we prove the Theorem 3.1.

Lemma 3.1 *Suppose that the hypotheses (a1) - (a4) hold. Then the Cauchy problem for the delay impulsive differential equation (1),(3),(8),(9) has a unique solution in $\mathbf{PAC}(\tau_1, \dots, \tau_k, \dots)$.*

Proof. Under the hypotheses (a1) -(a4) there exists one and only one solution of the delay differential equation (1),(3) on the interval $[0, \tau_1)$ [23]. The solution of the impulsive equation (1),(3) on the interval $[\tau_1, \tau_2)$ can be treated as the Cauchy problem for the delay differential equation (without impulses) on this interval with the initial value

$$x(\tau_1) = B_1 x(\tau_1 - 0) + \alpha_1$$

and the initial function

$$\varphi_1(s) = \varphi(s), s < 0, \varphi_1(s) = x(s), s \in [0, \tau_1).$$

This initial function is bounded and Borel measurable since φ possesses this property and x is continuous and bounded on $[0, \tau_1)$. By induction the solution can be constructed on the whole semi-axis $[0, \infty)$ and it is obviously unique. The proof of the lemma is complete.

In the stability theory for delay differential equations the "s-curtailed" theorem is used: if $C(t, s)$ is the Cauchy matrix of the equation (1), then $C(\cdot, s)$ is the solution of the Cauchy problem (6),(7), $x(s) = E_n$.

For impulsive delay differential equations the same statement is valid. First an auxiliary assertion will be proven.

Lemma 3.2 *For the fundamental matrix of "s-curtailed" equation the following estimate is valid*

$$\| X(t, s) \| \leq \prod_{s < \tau_i \leq t} (1 + \| B_i \|) \exp \left\{ \int_s^t \sum_{k=1}^m \| A_k(\zeta) \| d\zeta \right\}.$$

Proof. Let $s < \tau_i < \tau_{i+1} < \dots < \tau_j \leq t$. Then for $t \in [s, \tau_i)$ the solution x of the problem (6),(7),(2), $x(s) = E_n$ can be presented as

$$x(t) = E_n - \int_s^t \sum_{k=1}^m A_k(\zeta) x(h_k(\zeta)) d\zeta.$$

Hence

$$\|x(t)\| \leq 1 + \int_s^t \sum_{k=1}^m \|A_k(\zeta)\| \sup_{\xi \in [s, \zeta]} \|x(\xi)\| d\zeta. \quad (11)$$

Denote $y(t) = \sup_{\zeta \in [s, t]} \|x(\zeta)\|$. For the function $y(t)$ the inequality (11) implies

$$y(t) \leq 1 + \int_s^t \sum_{k=1}^m \|A_k(\zeta)\| y(\zeta) d\zeta.$$

Then the Gronwall - Bellman inequality gives

$$y(t) \leq \exp \left\{ \int_s^t \sum_{k=1}^m \|A_k(\zeta)\| d\zeta \right\}.$$

Therefore for the solution x of the problem (6),(7),(2), $x(s) = E_n$ we have obtained the estimate

$$\|x(t)\| \leq \exp \left\{ \int_s^t \sum_{k=1}^m \|A_k(\zeta)\| d\zeta \right\}. \quad (12)$$

Let $\tau_i \leq t < \tau_{i+1}$. Then

$$x(t) = x(\tau_i) - \int_{\tau_i}^t \sum_{k=1}^m A_k(\zeta) x(h_k(\zeta)) d\zeta.$$

Thus the inequality (12) and the impulsive condition $x(\tau_i) = B_i x(\tau_i - 0)$ imply the estimate

$$\begin{aligned} \|x(t)\| &\leq (1 + \|B_i\|) \exp \left\{ \int_s^{\tau_i} \sum_{k=1}^m \|A_k(\zeta)\| d\zeta \right\} + \\ &+ \int_{\tau_i}^t \sum_{k=1}^m \|A_k(\zeta)\| \|x[h_k(\zeta)]\| d\zeta. \end{aligned}$$

Hence if we again denote $y(t) = \max_{\zeta \in [s, t]} \|x(\zeta)\|$ we obtain

$$\begin{aligned} y(t) &\leq (1 + \|B_i\|) \exp \left\{ \int_s^{\tau_i} \sum_{k=1}^m \|A_k(\zeta)\| d\zeta \right\} + \\ &\quad + \int_{\tau_i}^t \sum_{k=1}^m \|A_k(\zeta)\| y(\zeta) d\zeta. \end{aligned}$$

Repeating the previous argument gives

$$\begin{aligned} \|x(t)\| &\leq (1 + \|B_i\|) \exp \left\{ \int_s^{\tau_i} \sum_{k=1}^m \|A_k(\zeta)\| d\zeta \right\} \exp \left\{ \int_{\tau_i}^t \sum_{k=1}^m \|A_k(\zeta)\| d\zeta \right\} = \\ &= (1 + \|B_i\|) \exp \left\{ \int_s^t \sum_{k=1}^m \|A_k(\zeta)\| d\zeta \right\}, \quad t \in [\tau_i, \tau_{i+1}). \end{aligned}$$

By considering the solution x of the problem (6),(7), $x(s) = E_n$ in the intervals $[\tau_{i+1}, \tau_{i+2}), \dots, [\tau_{j-1}, \tau_j)$ and at the point τ_j we obtain the required inequality for $\|X(t, s)\|$. This completes the proof of the lemma.

Remark. If $B_i \neq 0, i = 1, 2, \dots$, then in the statement of the lemma one can write $\prod \|B_i\|$ instead of $\prod(1 + \|B_i\|)$.

Corollary. For any $b > 0$ the function $X(t, s)$ is bounded in $[0, b] \times [0, b]$.

This result immediately follows from the inequality

$$\|X(t, s)\| \leq \prod_{0 < \tau_i \leq b} (\|B_i\| + 1) \exp \left\{ \int_0^b \sum_{k=1}^m \|A_k(\zeta)\| d\zeta \right\},$$

where $t \leq b$.

Now we can prove the "s - curtailment" result for impulsive delay differential equations.

Lemma 3.3 *Let $X(t, s)$ be the fundamental matrix of the "s -curtailed" equation . Then the solution y of the Cauchy problem (1),(2) , $y(\xi) = 0, \xi < 0, y(0) = 0$ can be presented as*

$$y(t) = \int_0^t X(t, s)r(s)ds. \quad (13)$$

Proof. We shall prove that (13) is the solution of the problem (1),(2) , $y(\xi) = 0, \xi < 0, y(0) = 0$. By differentiating the equality (13) in t we obtain

$$\dot{y}(t) = r(t) + \int_0^t X'_t(t, s)r(s)ds, \quad (14)$$

since $X(t, t) = E_n$. The equality (13) implies

$$\begin{aligned} y[h_k(t)] &= \int_0^{h_k^+(t)} X(h_k(t), s)r(s)ds = \\ &= \int_0^t X(h_k(t), s)r(s)ds - \int_{h_k^+(t)}^t X(h_k(t), s)r(s)ds, \end{aligned}$$

where $h^+ = \max\{h, 0\}$.

As $X(t, s) = 0$ for $t < s$ then the second integral in the right-hand side vanishes. Hence

$$y(h_k(t)) = \int_0^t X(h_k(t), s)r(s)ds,$$

that together with the equality (14) gives that y is the solution of the problem (1) , $y(\xi) = 0, \xi < 0, y(0) = 0$.

It remains to show that y satisfies the impulsive conditions

$$y(\tau_i) = B_i y(\tau_i - 0).$$

Let i be a fixed positive integer and $\{t_k\}_{k=1}^\infty \subset [0, \tau_i)$ be such sequence that t_k tends to τ_i as $k \rightarrow \infty$. We shall prove that the equality

$$\lim_{t_k \rightarrow \tau_i - 0} \int_0^{t_k} X(t_k, s)r(s)ds = \int_0^{\tau_i} X(\tau_i - 0, s)r(s)ds \quad (15)$$

holds, i.e that the limit under the integral is possible.

Denote

$$g_k(s) = X(t_k, s)\chi_{[0, t_k]}(s)r(s),$$

$$g(s) = X(\tau_i - 0, s)\chi_{[0, \tau_i)}(s)r(s).$$

Evidently

$$X(t_k, s) \rightarrow X(\tau_i - 0, s)$$

and

$$|\chi_{[0, t_k)}(s) - \chi_{[0, \tau_i)}(s)| = \chi_{(t_k, \tau_i)} \rightarrow 0$$

as $k \rightarrow \infty$ for any $s \geq 0$.

Therefore $\lim_{k \rightarrow \infty} g_k(s) = g(s)$, $s \geq 0$. Besides this, Lemma 3.2 gives (see the corollary)

$$\|g_k(s)\| \leq \prod_{0 < j < i} (\|B_j\| + 1) \exp \left\{ \int_0^{\tau_i} \sum_{k=1}^m \|A_k(\zeta)\| d\zeta \right\} \|r(s)\|.$$

Therefore the functions $g_k(s)$ are uniformly bounded for $s \leq \tau_i$.

By the Lebesgue theorem on limit under the integral we obtain (15). The function $X(t, s)$ satisfies the impulsive condition $X(\tau_i, s) = B_i X(\tau_i - 0, s)$. Thus the equality (15) implies

$$\begin{aligned} B_i y(\tau_i - 0) &= B_i \lim_{t_k \rightarrow \tau_i - 0} \int_0^{t_k} X(t_k, s) r(s) ds = \\ &= \int_0^{\tau_i} B_i X(\tau_i - 0, s) r(s) ds = \int_0^{\tau_i} X(\tau_i, s) r(s) ds = y(\tau_i). \end{aligned}$$

Hence $y(\tau_i) = B_i y(\tau_i - 0)$, which completes the proof of the lemma.

Proof of Theorem 3.1. By Lemma 3.1 there exists a unique solution of the Cauchy problem (we notice that the sum $\sum_{j=0}^{\infty} X(t, \tau_j) \alpha_j$ is definite since for each $t > 0$ this sum contains only a finite number of terms with $\tau_j \leq t$). By the direct substitution one can be convinced that the solution of the problem

$$\begin{aligned} \dot{x}(t) + \sum_{k=1}^m A_k(t) x[h_k(t)] &= r(t) - \sum_{k=1}^m A_k(t) \varphi[h_k(t)], \\ x(\xi) &= 0, \quad \xi < 0, \quad \varphi(\zeta) = 0, \quad \zeta \geq 0, \end{aligned} \tag{16}$$

coincides with the solution of the problem (1), $x(0) = 0$.

Lemma 3.3 gives that the solution of the problem (16),(2), $x(\xi) = 0, \xi \leq 0$, $x(0) = 0$ can be presented as

$$x_1(t) = \int_0^t X(t, s)r(s)ds - \sum_{i=1}^m \int_0^t X(t, s)A_i(s)\varphi(h_i(s))ds.$$

Define $x_2(t) = \sum_{j=0}^{\infty} X(t, \tau_j)\alpha_j$.

Since $X(t, s)$ is the fundamental matrix of the "s- curtailed" equation then

$$X'_t(t, \tau_j)\alpha_j + \sum_{i=1}^m A_i(t)X(t, h_i(\tau_j))\alpha_j = 0, \quad j = 1, 2, \dots$$

Thus

$$\sum_{j=0}^{\infty} X'_t(t, \tau_j)\alpha_j + \sum_{j=0}^{\infty} \sum_{i=1}^m A_i(t)X(t, h_i(\tau_j))\alpha_j = 0.$$

Therefore $x = x_1 + x_2$ satisfies the equation (16).

It remains to show that the initial condition and the impulsive conditions (9) are satisfied.

For any $s \geq 0, j = 1, 2, \dots$ we have (see the proof of Lemma 3.3)

$$X(\tau_j, s) = B_j X(\tau_j - 0, s)$$

and

$$\int_0^{\tau_j} X(\tau_j, s)g(s)ds = B_j \int_0^{\tau_j} X(\tau_j - 0, s)g(s)ds$$

for any $g \in \mathbf{L}_{\infty}$.

Hence

$$\begin{aligned} x(\tau_j) &= \int_0^{\tau_j} X(\tau_j, s)r(s)ds - \\ &\quad - \sum_{i=1}^m \int_0^{\tau_j} X(\tau_j, s)A_i(s)\varphi(h_i(s))ds + \sum_{k=0}^j X(\tau_j, \tau_k)\alpha_k = \\ &= B_j \int_0^{\tau_j} X(\tau_j - 0, s)r(s)ds - \sum_{i=1}^m B_j \int_0^{\tau_j} X(\tau_j - 0, s)A_i(s)\varphi(h_i(s))ds + \\ &\quad + B_j \sum_{k=0}^{j-1} X(\tau_j - 0, \tau_k)\alpha_k + X(\tau_j, \tau_j)\alpha_j = \\ &= B_j \left[\int_0^{\tau_j} X(\tau_j - 0, s)r(s)ds - \sum_{i=1}^m \int_0^{\tau_j} X(\tau_j - 0, s)A_i(s)\varphi(h_i(s))ds + \right. \end{aligned}$$

$$+ \sum_{k=0}^{j-1} X(\tau_j - 0, \tau_k) \alpha_k] + E_n \alpha_j = B_j x(\tau_j - 0) + \alpha_j,$$

i.e the impulsive conditions (9) are satisfied.

Further,

$$x(0) = \sum_{i=0}^{\infty} X(0, \tau_i) \alpha_i = X(0, 0) \alpha_0 = \alpha_0$$

since $X(t, s) = 0$, $t < s$. Therefore the initial condition (8) is also satisfied. The proof of the theorem is complete.

Definition 4. The operator $C : \mathbf{L}_{\infty} \rightarrow \mathbf{PAC}(\tau_1, \dots, \tau_k, \dots)$ defined by the formula

$$(Cf)(t) = \int_0^t X(t, s) f(s) ds$$

is said to be **the Cauchy operator** of the impulsive delay differential equation (1),(2),(3).

4 Exponential estimates of the Cauchy matrix

The purpose of this work is to obtain the exponential estimate of the matrix $X(t, s)$ and to investigate the exponential stability of the impulsive delay differential equations.

Let $\mathbf{D}_{\infty} \subset \mathbf{PAC}(\tau_1, \dots, \tau_k, \dots)$ be a space of functions $x : [0, \infty) \rightarrow \mathbf{R}^n$ absolutely continuous on the intervals $[\tau_j, \tau_{j+1})$ satisfying the impulsive conditions (2) and such that both x and its derivative are essentially bounded on $[0, \infty)$.

We introduce the norm in \mathbf{D}_{∞}

$$\|x\|_{\mathbf{D}_{\infty}} = \|x\|_{\mathbf{L}_{\infty}} + \|\dot{x}\|_{\mathbf{L}_{\infty}}.$$

The main result of this section is the following.

Theorem 4.1 *Let exist constant $\delta > 0$ such that*

$$\theta_i(t) = t - h_i(t) \leq \delta, t \geq 0,$$

and the hypotheses (a1) – (a6) hold. Suppose that for any essentially bounded on $[0, \infty)$ right-hand side r all solutions of the impulsive equation (1),(2),(3) with $\varphi \equiv 0$, are essentially bounded on $[0, \infty)$ together with their derivatives.

Then there exist positive constants N and ν such that for the Cauchy matrix of the impulsive equation (1),(2),(3) the inequality

$$\| X(t, s) \| \leq N \exp[-\nu(t - s)]$$

holds.

For proving this theorem we need some auxiliary assertions.

Lemma 4.1 *Suppose the hypotheses (a5) and (a6) hold, i.e.*

$$M = \sup_i \| B_i \| < \infty \text{ and } I = \lim_{b \rightarrow \infty} \sup_{t, s > b} \frac{i(t, s)}{t - s} < \infty,$$

where $i(t, s)$ is a number of the points τ_j belonging to the segment $[s, t]$. Then D_∞ is a Banach space.

Proof. We choose a positive constant a such that $I \ln M < a$. Then for the linear impulsive equation $\dot{x} + ax = f$, (2) the Cauchy matrix $C_0(t, s)$ has an exponential estimate. For proving this we use the representation [2,3]

$$C_0(t, s) = \exp[-a(t - s)] \prod_{s < \tau_i \leq t} B_i.$$

Thus the following estimate is valid

$$\| C_0(t, s) \| \leq \exp \left\{ - \left[a - \frac{\ln M \cdot i(t, s)}{t - s} \right] (t - s) \right\} \leq \exp[-\nu(t - s)],$$

where $\nu = a - I \ln M$.

Denote

$$(\mathcal{L}x)(t) = \dot{x} + ax,$$

with $x(0) = 0$, $x(\tau_i) = B_i x(\tau_i - 0)$,

$$(Cf)(t) = \int_0^t C_0(t, s)f(s)ds.$$

Thus we have introduced the linear operators $\mathcal{L} : \mathbf{D}_\infty \rightarrow \mathbf{L}_\infty$ and $C : \mathbf{L}_\infty \rightarrow \mathbf{D}_\infty$. Now we shall prove that these operators are continuous.

To this end

$$\| \mathcal{L}x \|_{\mathbf{L}_\infty} = \| \dot{x} + ax \|_{\mathbf{L}_\infty} \leq (1 + a) \| x \|_{\mathbf{D}_\infty},$$

$$\| Cf \|_{\mathbf{D}_\infty} = \| x \|_{\mathbf{L}_\infty} + \| \dot{x} \|_{\mathbf{L}_\infty},$$

where

$$\begin{aligned} \| x \|_{\mathbf{L}_\infty} &\leq \frac{1}{\nu} \| f \|_{\mathbf{L}_\infty}, \\ \| \dot{x} \|_{\mathbf{L}_\infty} &= \| f - ax \|_{\mathbf{L}_\infty} \leq \\ &\leq \| f \|_{\mathbf{L}_\infty} + a \frac{1}{\nu} \| f \|_{\mathbf{L}_\infty} = \left(1 + \frac{a}{\nu}\right) \| f \|_{\mathbf{L}_\infty}, \end{aligned}$$

therefore

$$\| Cf \|_{\mathbf{D}_\infty} \leq \left(1 + \frac{1+a}{\nu}\right) \| f \|_{\mathbf{L}_\infty}.$$

Now we prove that the space \mathbf{D}_∞ is complete. Let $\{x_k\}_{k=1}^\infty \subset \mathbf{D}_p$ be a sequence such that $\| x_k - x_i \|_{\mathbf{D}_\infty} \rightarrow 0$ as $k, i \rightarrow \infty$. Consider the functions $f_k = \mathcal{L}x_k$. Then

$$x_k(t) = C_0(t, 0) + \int_0^t C_0(t, s)f_k(s)ds,$$

where $C_0(t, 0)$ is the solution of the problem

$$\dot{x} + ax = 0, \quad x(0) = E_n, \quad x(\tau_i) = B_i x(\tau_i - 0).$$

First we show that the sequence $\{g_k(t)\}$, with $g_k(t) = C_0(t, 0)x_k(0)$, converges in \mathbf{D}_∞ . Let $b < \tau_1$. Then $\| x_k - x_i \|_{\mathbf{D}_\infty} \rightarrow 0$ as $k, i \rightarrow \infty$ implies

$$\max_{t \in [0, b]} [\| x_k(t) - x_i(t) \|] + \text{vraisup}_{t \in [0, b]} [\| \dot{x}_k(t) - \dot{x}_i(t) \|] \rightarrow 0.$$

Therefore

$$\| x_k(0) - x_i(0) \| \rightarrow 0.$$

Since \mathbf{R}^n is complete then there exists β such that $x_k(0) \rightarrow \beta$ as $k \rightarrow \infty$.

The estimates $\|C_0(t, 0)\| \leq \exp(-\nu t)$ and $\|\dot{C}_0(t, 0)\| \leq a \exp(-\nu t)$ imply that $g_k(t) \rightarrow C_0(t, 0)\beta$ in \mathbf{D}_∞ .

Denote

$$\tilde{x}_k = \int_0^t C_0(t, s) f_k(s) ds.$$

Then

$$\|\tilde{x}_k - \tilde{x}_i\|_{\mathbf{D}_\infty} = \left\| \int_0^t C_0(t, s) [f_k(s) - f_i(s)] ds \right\|_{\mathbf{D}_\infty} \rightarrow 0$$

implies

$$\|f_k - f_i\|_{\mathbf{L}_\infty} = \|\mathcal{L}(x_k - x_i)\|_{\mathbf{L}_\infty} \rightarrow 0$$

since $\mathcal{L} : \mathbf{D}_\infty \rightarrow \mathbf{L}_\infty$ is a continuous operator.

As \mathbf{L}_∞ is complete then there exists $f \in \mathbf{L}_\infty$ such that

$$\lim_{k \rightarrow \infty} \|f_k - f\|_{\mathbf{L}_\infty} = 0.$$

Let $\tilde{x} = Cf$. Since C is a continuous operator, then

$$\|\tilde{x}_k - \tilde{x}\|_{\mathbf{D}_\infty} = \|C(f_k - f)\|_{\mathbf{D}_\infty} \rightarrow 0.$$

Thus $\tilde{x} = \lim_{k \rightarrow \infty} \tilde{x}_k \in \mathbf{D}_\infty$.

Let

$$x = C_0(t, 0)\beta + \tilde{x}.$$

Then x_k tends to x in \mathbf{D}_∞ , which completes the proof of the lemma.

Lemma 4.2 *Let $\mathcal{L}, \mathcal{M} : \mathbf{D}_\infty \rightarrow \mathbf{L}_\infty$ be linear bounded operators and $C, C_{\mathcal{M}}$ be the Cauchy operator of the equations $\mathcal{L}x = f, \mathcal{M}x = f$ correspondingly. Suppose the Cauchy operator C is a bounded operator acting from \mathbf{L}_∞ to \mathbf{D}_∞ and the operator $\mathcal{M}C : \mathbf{L}_\infty \rightarrow \mathbf{L}_\infty$ is invertible. Then the Cauchy operator $C_{\mathcal{M}}$ also acts from \mathbf{L}_∞ to \mathbf{D}_∞ and is bounded.*

Proof. For each $f \in \mathbf{L}_\infty$ the function $x = C(\mathcal{M}C)^{-1}f$ is the solution of the problem $\mathcal{M}x = f, x(0) = 0$. By the definition of the Cauchy operator

$C_{\mathcal{M}} = C(\mathcal{M}C)^{-1}$. This implies that $C_{\mathcal{M}}$ is a bounded operator acting from \mathbf{L}_{∞} to \mathbf{D}_{∞} , which completes the proof of the lemma.

Proof of Theorem 4.1. First we shall obtain an exponential estimate for $X(t, 0)$. Let ν be a positive number. Denote

$$y(t) = x(t) \exp(\nu t).$$

Everywhere below we assume $x(\xi) = y(\xi) = 0, \xi < 0$. If $x(t)$ satisfies the impulsive conditions (2), then evidently

$$y(\tau_j) = B_j y(\tau_j - 0), \quad j = 1, 2, \dots$$

We denote

$$(\mathcal{L}x)(t) = \dot{x}(t) + \sum_{i=1}^m A_i(t)x[h_i(t)].$$

By substituting $x(t) = y(t) \exp(-\nu t)$ we obtain

$$\begin{aligned} (\mathcal{L}x)(t) &= \exp(-\nu t) \dot{y}(t) - \nu \exp(-\nu t) y(t) + \\ &+ \sum_{i=1}^m \exp(-\nu h_i(t)) A_i(t) y[h_i(t)] = \exp(-\nu t) \left\{ \dot{y}(t) + \sum_{i=1}^m A_i(t) y[h_i(t)] + \right. \\ &+ \sum_{i=1}^m \exp[\nu(t - h_i(t))] A_i(t) y[h_i(t)] - \sum_{i=1}^m A_i(t) y[h_i(t)] - \nu y(t) \left. \right\} = \\ &= \exp(-\nu t) \left\{ (\mathcal{L}y)(t) - \nu y(t) + \sum_{i=1}^m [\exp(\nu \theta_i(t)) - 1] A_i(t) y[h_i(t)] \right\}. \end{aligned}$$

Denote

$$\begin{aligned} (\mathcal{T}y)(t) &= \sum_{i=1}^m [\exp(\nu \theta_i(t)) - 1] A_i(t) y[h_i(t)] - \nu y(t), \\ (\mathcal{M}y)(t) &= (\mathcal{L}y)(t) + (\mathcal{T}y)(t). \end{aligned}$$

Then

$$(\mathcal{L}x)(t) = \exp(-\nu t) (\mathcal{M}y)(t).$$

The impulsive equation (1),(2),(3) has a solution that can be presented as (13). Let C be the Cauchy operator of the equation (1),(2).

Under the hypotheses of the theorem the operator $\mathcal{L} : \mathbf{D}_\infty \rightarrow \mathbf{L}_\infty$ is bounded. Since a solution is in \mathbf{L}_∞ together with its derivative for any right-hand side from \mathbf{L}_∞ then the Cauchy operator C acts from \mathbf{L}_∞ to \mathbf{D}_∞ .

Denote $\mathbf{D}_\infty^0 = \{x \in \mathbf{D}_\infty : x(0) = 0\}$. Let \mathcal{L}^0 be the contraction of \mathcal{L} to \mathbf{D}_∞^0 . Then the operator $\mathcal{L}^0 : \mathbf{D}_\infty^0 \rightarrow \mathbf{L}_\infty$ acts onto the space \mathbf{L}_∞ . Therefore by the Banach theorem the Cauchy operator $C : \mathbf{L}_\infty \rightarrow \mathbf{D}_\infty^0$ is bounded as its inverse.

By (a2) the columns of A_i belong to \mathbf{L}_∞ , i.e. there exists $Q > 0$ such that

$$\text{vraisup}_{t \geq 0} \sum_{i=1}^m \|A_i(t)\| < Q.$$

Therefore the operator $\mathcal{T} : \mathbf{D}_\infty \rightarrow \mathbf{L}_\infty$ is bounded, with

$$\|\mathcal{T}\|_{\mathbf{D}_\infty \rightarrow \mathbf{L}_\infty} \leq Q[\exp(\nu\delta) - 1] + \nu.$$

Thus $\mathcal{M}C = \mathcal{L}C + \mathcal{T}C = E + \mathcal{T}C$ has a bounded inverse operator whenever

$$\|\mathcal{T}C\|_{\mathbf{L}_\infty \rightarrow \mathbf{L}_\infty} < 1.$$

Here E is the identity operator.

Let $P = \|C\|_{\mathbf{L}_\infty \rightarrow \mathbf{D}_\infty}$. Then

$$\|\mathcal{T}C\|_{\mathbf{L}_\infty \rightarrow \mathbf{L}_\infty} \leq P \|\mathcal{T}\|_{\mathbf{D}_\infty \rightarrow \mathbf{L}_\infty} \leq PQ[\exp(\nu\delta) - 1] + P\nu < 1$$

for ν being small enough.

The operators \mathcal{L} and \mathcal{T} continuously act from \mathbf{D}_∞ to \mathbf{L}_∞ . Therefore the operator $\mathcal{M} = \mathcal{L} - \mathcal{T}$ also continuously acts from \mathbf{D}_∞ to \mathbf{L}_∞ . Hence Lemma 4.2 implies that the Cauchy operator $C_{\mathcal{M}}$ of the equation $\mathcal{M}y = f$, $x(0) = 0$ continuously acts from \mathbf{L}_∞ into \mathbf{D}_∞ .

We introduce

$$\Psi_0(t) = \begin{cases} E_n, & \text{if } t \leq \tau_1, \\ \prod_{0 < \tau_j \leq t} B_j, & \text{if } t > \tau_1, \ M \leq 1, \\ \exp\{-\ln M(I + \varepsilon)t\} \prod_{0 < \tau_j \leq t} B_j, & \text{if } t > \tau_1, \ M > 1, \end{cases}$$

where ε is a positive constant, I and M are the numbers defined in the hypotheses (a5), (a6).

Let us prove that columns of both $\Psi_0(t)$ and its derivative are in \mathbf{L}_∞ .

Let $M \leq 1$. Evidently $\|\Psi_0(t)\| \leq 1$ and $\dot{\Psi}_0(t) = 0$, therefore in the case $M \leq 1$ the columns of Ψ and $\dot{\Psi}$ are in \mathbf{L}_∞ .

Let $M > 1$. By the definition of the function $i(t, s)$ for any fixed $\varepsilon > 0$ there exists $b > 0$ such that

$$I > \frac{i(t, s)}{t - s} - \varepsilon$$

holds for any $t, s \geq b$.

Therefore for $t > b$

$$\begin{aligned} \|\Psi_0(t)\| &\leq \prod_{0 < \tau_i \leq b} \|B_i\| \exp \left[\sum_{b < \tau_i \leq t} \ln \|B_i\| \right] \exp[-\ln M(I + \varepsilon)t] \leq \\ &\leq \prod_{0 < \tau_i \leq b} \|B_i\| \exp[i(t, b) \ln M] \exp[-\ln M(I + \varepsilon)t] \leq \\ &\leq \prod_{0 < \tau_i \leq b} \|B_i\| \exp[\ln M(I + \varepsilon)(t - b)] \exp[-\ln M(I + \varepsilon)t] = \\ &= \prod_{0 < \tau_i \leq b} \|B_i\| \exp[-\ln M(I + \varepsilon)b] \leq \prod_{0 < \tau_i \leq b} \|B_i\|. \end{aligned}$$

Thus $\Psi_0(t)$ is in \mathbf{L}_∞ since

$$\|\Psi_0(t)\| \leq \prod_{0 < \tau_i \leq b} (\|B_i\| + 1)$$

for $t \leq b$. As

$$\dot{\Psi}_0(t) = -\ln M(I + \varepsilon)\Psi_0(t)$$

for almost all t , then the derivative of Ψ_0 is also essentially bounded on $[0, \infty)$.

By construction Ψ_0 also satisfies the impulsive conditions

$$\Psi_0(\tau_i) = B_i \Psi_0(\tau_i - 0).$$

Thus the columns of Ψ_0 belong to \mathbf{D}_∞ . Hence the columns of $\mathcal{M}\Psi_0$ belong to \mathbf{L}_∞ .

If $Y(t, 0)$ is the solution of the homogeneous impulsive equation $\mathcal{M}y = 0$, (2), $y(0) = E_n$, then $z(t) = \Psi_0(t) - Y(t, 0)$ is the solution of the problem

$$\mathcal{M}z = \mathcal{M}\Psi_0, \quad z(\tau_i) = B_i(\tau_i - 0), \quad z(0) = 0.$$

Therefore by Lemma 3.3

$$Y(t, 0) = \Psi_0(t) - \int_0^t C_{\mathcal{M}}(t, s)(\mathcal{M}\Psi_0)(s)ds.$$

The operator $C_{\mathcal{M}}$ continuously acts from \mathbf{L}_{∞} into \mathbf{D}_{∞} and the columns of $\mathcal{M}\Psi_0$ belong to \mathbf{L}_{∞} . Hence the columns of $Y(t, 0)$ belong to \mathbf{D}_{∞} , therefore $Y(t, 0)$ is bounded on the semi-axis $[0, \infty)$

$$\| Y(t, 0) \| \leq N_0, \quad t \geq 0.$$

Thus the equality $X(t, 0) = \exp(-\nu t)Y(t, 0)$ implies

$$\| X(t, 0) \| \leq N_0 \exp(-\nu t), \quad t \geq 0.$$

Now we shall prove that the same estimate is valid for the Cauchy matrix $X(t, s)$. The matrix $X(t, s)$ is the solution of the "s - curtailed" equation (6),(7),(2), therefore by repeating the proof for this equation one easily obtains

$$\| X(t, s) \| \leq N_s \exp[-\nu_s(t - s)], \quad t \geq s.$$

It remains to show that N and ν can be chosen independently of s . The space \mathbf{L}_{∞} contains functions vanishing on $[0, s)$, therefore

$$\| C \|_{\mathbf{L}_{\infty}([s, \infty)) \rightarrow \mathbf{D}_{\infty}([s, \infty))} \leq P.$$

The constant Q also does not depend on s . Thus

$$\| \mathcal{T}C \|_{\mathbf{L}_{\infty}([s, \infty)) \rightarrow \mathbf{L}_{\infty}([s, \infty))} \leq PQ[\exp(\nu\delta) - 1] + P\nu = q < 1$$

for the same constant ν . Here $s \geq 0$ is arbitrary and the values ν, q do not depend on s . If ν is chosen small enough for $q < 1$ then the inverse operator in \mathbf{L}_{∞}

$$(E + \mathcal{T}C)^{-1}$$

exists and

$$\| (E + \mathcal{T}C)^{-1} \|_{\mathbf{L}_{\infty}([s, \infty)) \rightarrow \mathbf{L}_{\infty}([s, \infty))} \leq \frac{q}{1 - q}$$

for any $s \geq 0$.

Hence (see the proof of Lemma 4.2)

$$\| C_{\mathcal{M}} \|_{\mathbf{L}_{\infty}([s, \infty)) \rightarrow \mathbf{D}_{\infty}([s, \infty))} \leq$$

$$\leq \| C \|_{\mathbf{L}_\infty([s,\infty)) \rightarrow \mathbf{D}_\infty([s,\infty))} \| (E - \mathcal{T}C)^{-1} \|_{\mathbf{L}_\infty([s,\infty)) \rightarrow \mathbf{L}_\infty([s,\infty))} .$$

Thus we have obtained

$$\| C_{\mathcal{M}} \|_{\mathbf{L}_\infty([s,\infty)) \rightarrow \mathbf{D}_\infty([s,\infty))} \leq \frac{qP}{1-q} \quad (17)$$

for any $s \geq 0$.

We introduce the functions $\Psi_s(t)$, $s \geq 0$, $t \geq s$

$$\Psi_s(t) = \begin{cases} E_n, & \text{if } \tau_i \notin (s, t], \ i = 1, 2, \dots \\ \prod_{s < \tau_j \leq t} B_j, & \text{if } M \leq 1, \exists \tau_i \in (s, t], \\ \exp\{-\ln M(I + \varepsilon)(t - s)\} \prod_{s < \tau_j \leq t} B_j, & \text{otherwise.} \end{cases}$$

As in the case $s = 0$ one easily obtains

$$\begin{aligned} \| \Psi_s(t) \| &\leq \prod_{s < \tau_i \leq b} (\| B_i \| + 1) \leq \\ &\leq \prod_{0 < \tau_i \leq b} (\| B_i \| + 1) = U \end{aligned}$$

for every $t \geq s \geq 0$. Hence $\Psi_s \in \mathbf{L}_\infty$. Similarly one obtains that the derivative of $\Psi_s(t)$ in t is in \mathbf{L}_∞ . Evidently Ψ_s satisfies the impulsive conditions (2). Therefore the columns of Ψ_s are in \mathbf{D}_∞ .

By Lemma 3.3 the Cauchy matrix $Y(t, s)$ of the impulsive equation

$$\mathcal{M}y = f, \ y(\tau_i) = B_i y(\tau_i - 0)$$

can be presented as

$$Y(t, s) = \Psi_s(t) - \int_s^t C_{\mathcal{M}}(t, s)(\mathcal{M}\Psi_s)(s)ds.$$

Therefore the inequality (17) implies

$$\| Y(t, s) \| \leq U + qPU \| \mathcal{M} \|_{\mathbf{D}_\infty \rightarrow \mathbf{L}_\infty} / (1 - q)$$

for any $t, s \geq 0$. Choosing

$$N = qPU \| \mathcal{M} \|_{\mathbf{D}_\infty \rightarrow \mathbf{L}_\infty} / (1 - q) + U$$

we obtain the required estimate

$$\| X(t, s) \| \leq N \exp(-\nu(t - s)).$$

The proof of the theorem is complete.

The following example shows that the bounded delay condition $t - h(t) < \delta$ in Theorem 4.1 is essential.

Example. The solution of the scalar equation

$$\dot{x}(t) + x(t) - x(0) = f(t)$$

is bounded for any bounded f but $x(t) \equiv 1$ is the solution of the corresponding homogeneous equation.

5 Exponential stability

Theorem 5.1 *Suppose the hypotheses of Theorem 4.1 hold. Then the impulsive equation (1),(2),(3) is exponentially stable .*

Proof. We immediately obtain the exponential stability with respect to the initial value from Theorems 3.1 and 4.1 since

$$\| x(t) \| = \| X(t, 0)x(0) \| \leq N \exp(-\nu t) \| x(0) \|^r$$

for $r \equiv \varphi \equiv 0$.

As $t - h_i(t) \leq \delta$ then

$$\varphi[h_i(t)] = 0, \quad t > \delta, \quad i = 1, \dots, m.$$

Therefore for $r \equiv 0, x(0) = 0$

$$\| x(t) \| = \left\| \int_0^t X(t, s) \sum_{i=1}^m A_i(s) \varphi[h_i(s)] ds \right\| =$$

$$\begin{aligned}
&= \left\| \int_0^\delta X(t, s) \sum_{i=1}^m A_i(s) \varphi[h_i(s)] ds \right\| \leq \\
&\leq \int_0^\delta \left\| X(t, s) \right\| \sum_{i=1}^m \left\| A_i(s) \right\| \left\| \varphi[h_i(s)] \right\| ds \leq \\
&\leq \sup_{t \geq 0} \sum_{i=1}^m \left\| A_i(t) \right\| \cdot \sup_{s < 0} \left\| \varphi(s) \right\| \cdot \int_0^\delta \exp[-\nu(t-s)] ds \leq \\
&\leq \sup_{t \geq 0} \sum_{i=1}^m \left\| A_i(t) \right\| \cdot \sup_{s < 0} \left\| \varphi(s) \right\| \cdot \exp(-\nu t) [\exp(\nu \delta) + 1].
\end{aligned}$$

Denoting

$$N_0 = \sup_{t \geq 0} \sum_{i=1}^m \left\| A_i(t) \right\| \cdot [\exp(\nu \delta) + 1]$$

we obtain

$$\left\| x(t) \right\| \leq N_0 \exp(-\nu t) \cdot \sup_{s < 0} \left\| \varphi(s) \right\|,$$

i.e. the equation (1),(2)(3) is exponentially stable with respect to the initial function. The proof of the theorem is complete.

Theorem 5.1 can be used for obtaining stability results. For instance the following assertion is valid.

Theorem 5.2 *Suppose the hypotheses (a1) – (a6) hold and there exist positive constants δ, ζ, ρ such that $t - h_k(t) < \delta$, $\zeta \leq \tau_{i+1} - \tau_i \leq \rho$, $i = 1, 2, \dots$,*

$$\gamma = \sup_i \left\| B_i \right\| < 1,$$

and

$$\sum_{k=1}^m \text{vraisup}_{t > 0} \left\| A_k(t) \right\| \left[\frac{1}{\alpha} \exp(-\alpha \rho) + \rho \right] < 1, \quad (18)$$

where

$$\alpha = -\frac{1}{\zeta} \ln \gamma. \quad (19)$$

Then all solutions of the impulsive equation (1),(2) are exponentially stable.

For proving this result we consider an auxiliary ordinary impulsive equation

$$\dot{x}(t) = f(t), \quad t \geq 0, \quad x(\tau_i) = B_i x(\tau_i - 0), \quad (20)$$

where $\lim_{i \rightarrow \infty} \tau_i = \infty$. Let $C_0(t, s)$ be a Cauchy matrix of (20). We need the following result for C_0 .

Lemma 5.1 *Suppose there exist positive constants ρ and ζ such that*

$$\zeta \leq \tau_{i+1} - \tau_i \leq \rho.$$

Then

$$\|C_0(t, s)\| \leq \begin{cases} \exp[-\alpha(t-s)], & \text{if } t-s > \rho, \\ 1, & \text{otherwise,} \end{cases}$$

and for any $t > p, z \in \mathbf{L}_\infty$ the following inequality holds

$$\left\| \int_0^t C_0(t, s) z(s) ds \right\|_{\mathbf{L}_\infty} \leq \left[\frac{1}{\alpha} \exp(-\alpha\rho) + \rho \right] \|z\|_{\mathbf{L}_\infty}. \quad (21)$$

Here α is defined by (19).

Proof. The Cauchy matrix $C_0(t, s)$ is the solution of the problem

$$\dot{x} = 0, \quad x(\tau_i) = B_i x(\tau_i - 0), \quad x(s) = E_n, \quad t \in [s, \infty).$$

Therefore

$$C_0(t, s) = \begin{cases} \prod_{s < \tau_i \leq t} B_i, & \text{if there exists } \tau_i \in (s, t], \\ E_n, & \text{otherwise.} \end{cases}$$

Let $t - s > \rho$. Since in the case there exists i such that $s < \tau_i \leq t$ then

$$\begin{aligned} \|C_0(t, s)\| &\leq \exp \left[\sum_{s < \tau_i \leq t} \ln \|B_i\| \right] \leq \\ &\leq \exp \left[(t-s) \frac{i(t, s)}{t-s} \ln(\sup_i \|B_i\|) \right] \leq \exp \left[\frac{1}{\zeta} (t-s) \ln \gamma \right] = \exp[-\alpha(t-s)]. \end{aligned}$$

If $t - s < \rho$ and there exists $\tau_i \in (s, t]$, then the above estimate of $\|C_0(t, s)\|$ is also valid. As $\ln \gamma < 0$, then $\|C_0(t, s)\| \leq 1$.

If there is no τ_i belonging to $(s, t]$, then $\|C_0(t, s)\| = 1$. Thus we have obtained the estimate of $C_0(t, s)$.

Hence for each $z \in \mathbf{L}_\infty$

$$\begin{aligned} & \left\| \int_0^t C_0(t, s) z(s) ds \right\| \leq \int_0^{t-\rho} \|C_0(t, s)\| \|z(s)\| ds + \\ & + \int_{t-\rho}^t \|C_0(t, s)\| \|z(s)\| ds \leq \left\{ \int_0^{t-\rho} \exp[-\alpha(t-s)] ds + \int_{t-\rho}^t ds \right\} \|z\|_{\mathbf{L}_\infty} = \\ & = \left[\frac{1}{\alpha} \exp(-\alpha\rho) - \frac{1}{\alpha} \exp(-\alpha t) + \rho \right] \|z\|_{\mathbf{L}_\infty} \leq \left[\frac{1}{\alpha} \exp(-\alpha\rho) + \rho \right] \|z\|_{\mathbf{L}_\infty}, \end{aligned}$$

which completes the proof of the lemma.

Proof of Theorem 5.2. First we shall make the remark that will be used below. For the exponential estimation of the Cauchy matrix one can assume $A_k(t) \equiv 0$ for $t < \rho$. In fact by Lemma 3.3 if coefficients of two equations (1),(2) distinguish only on the finite segment $[0, b]$ then their Cauchy matrices coincide for $s > b$. Therefore either both of these matrices or none of them have an exponential estimate.

The proof is based on Theorem 5.1. Precisely, we shall establish that for any $f \in \mathbf{L}_\infty$ all solutions of the problem

$$\dot{x}(t) + \sum_{k=1}^m A_k(t)x[h_k(t)] = f(t),$$

$$x(\tau_i) = B_i x(\tau_i - 0)$$

$$x(0) = 0,$$

$$x(\xi) = 0, \text{ if } \xi < 0,$$

are in \mathbf{D}_∞ .

Let $C_0(t, s)$ be the Cauchy matrix of the equation (20). Then by substituting

$$x(t) = \int_0^t C_0(t, s) z(s) ds, \quad \dot{x} = z \tag{22}$$

we obtain the equation

$$z(t) + \sum_{k=1}^m \int_0^{h_k^+(t)} A_k(t) C_0(h_k(t), s) z(s) ds = f(t). \quad (23)$$

We shall prove that the equation (23) has a solution in \mathbf{L}_∞ for any $f \in \mathbf{L}_\infty$. To this end we consider the operator

$$(Hz)(t) = \sum_{k=1}^m \int_0^{h_k^+(t)} A_k(t) C_0(h_k(t), s) z(s) ds$$

and estimate its norm in \mathbf{L}_∞ .

Lemma 5.1 and the remark made in the beginning of the proof imply

$$\|H\|_{\mathbf{L}_\infty \rightarrow \mathbf{L}_\infty} \leq \sum_{k=1}^m \sup_{t>0} \|A_k(t)\| \left[\frac{1}{\alpha} \exp(-\alpha\rho) + \rho \right].$$

By the inequality (18) $\|H\|_{\mathbf{L}_\infty \rightarrow \mathbf{L}_\infty} < 1$. Hence for any $f \in \mathbf{L}_\infty$ the solution z of the equation (23) is in \mathbf{L}_∞ . By (22) and (21) we obtain that $x \in \mathbf{L}_\infty$ and $\dot{x} = z \in \mathbf{L}_\infty$. Thus $x \in \mathbf{D}_\infty$.

Applying Theorem 5.1 completes the proof of the theorem.

Remark. One can easily see that under the hypotheses of Theorem 5.2 the corresponding equation without impulses may be unstable. Thus Theorem 5.2 can be treated as a stabilization scheme (see also [8]).

Example. The scalar impulsive equation

$$\begin{aligned} \dot{x} - ax(t-h) &= f, \quad a > 0, \\ x(i) &= bx(i-1), \quad i = 1, 2, \dots, \end{aligned}$$

is exponentially stable if $|b| < 1$ and

$$a \left(1 - \frac{|b|}{\ln |b|} \right) < 1.$$

At the same time the corresponding delay differential equation

$$\dot{x} - ax(t-h) = f$$

is not stable.

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